

Thermalization in Yang-Mills Classical Mechanics

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Abstract

We use quartic oscillators system with two degrees of freedom to model Yang-Mills classical mechanics. This simple model explains qualitatively many features reported in lattice calculation of $(3+1)$ - dimensional classical Yang-Mills system. The largest Lyapunov exponent (λ) and the thermalization time were numerically evaluated. We also show, in our model, that λ scales with 4th root of energy density. Here thermalization is due to relaxation phenomena associated with the color degrees of freedom. From the physical picture of thermalization, we speculate that the system with coherent fields (flux tubes) formed in relativistic heavy ion collisions can relax by chaos and the estimated thermalization time can be smaller than $1 fm/c$.

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Quantum chromodynamics (QCD) is a Yang-Mills (YM) theory, nonlinear and hence exhibits chaos [1, 2]. This property is used to understand the confinement as well as deconfinement problems in QCD. Savvidy [1] extensively studied chaos in a homogeneous $SU(2)$ Yang-Mills system and speculated that it might be related to the confinement problem. Recently, Müller *et al.* [3] studied the classical YM system in $(3 + 1)$ - dimension on a lattice and showed that it exhibits deterministic chaos. They related YM chaos to the thermalization of gluons in the deconfined state, which may be formed in relativistic heavy ion collisions (RHICs). They evaluated the maximum Lyapunov exponent, λ , and found that it scales with energy per plaquette and estimated the thermalization time of QGP formed in RHICs. But their analysis involves heavy numerical computations. It was also not clear why λ linearly depends on energy per plaquette. In the light of their results, it is interesting to reanalyse the earlier work on Yang-Mills classical mechanics (YMCM), studied by Savvidy.

In this letter we study this problem using a very simple model, quartic oscillators (QO), which is related to YMCM of $SU(2)$ under certain limit. The Lyapunov exponent (LE) of QO was studied by Joy and Sabir [4] and its statistical mechanics and thermodynamics was studied by us [5]. Here we find that maximum LE, λ , scales with 4th root of energy density, which can be seen both by a dimensional argument as well as numerically and thermalization time is estimated. This is similar to the result of Müller *et al.*, but obtained with a simpler model, less numerical computations and with a clear physical picture of thermalization based on the statistical mechanics of chaotic system. The coherent field energy, like flux tubes formed in rela-

tivistic heavy ion collisions, may be redistributed among various degrees of freedom associated with spatial, color, chromo electric and magnetic fields due to chaos in YM theory.

The YMCM Hamiltonian is a $SU(2)$ YM Hamiltonian density in temporal gauge, $A_0^a = 0$, and assuming that the vector potentials $\vec{A}^a(t)$ are functions of time only. It is given by

$$H_{YM} = \frac{1}{2} \sum_a \dot{\vec{A}}_a^2 + \frac{1}{4} g^2 \sum_{a,b} (\vec{A}_a \times \vec{A}_b)^2, \quad (1)$$

where g is the gauge coupling constant and $a = 1, 2, 3$ are color quantum numbers. The Hamiltonian, H_{YM} , may be rewritten in the form

$$H_{YM} = H_{FS} + T_{YM} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}g^2(x^2y^2 + y^2z^2 + z^2x^2) + T_{YM}, \quad (2)$$

where H_{FS} is called a fundamental subsystem (FS) of YMCM and T_{YM} describes quasi-rotational freedoms. Let us consider a simpler two dimensional model ($a = 1, 2$ in Eq. (1)) and also without T_{YM} . The corresponding Hamiltonian is

$$H_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}g^2x^2y^2. \quad (3)$$

On redefining variables $X_1 \equiv gx$ and $X_2 \equiv gy$, it reduces to

$$H \equiv g^2 H_2 = \frac{1}{2}(\dot{X}_1^2 + \dot{X}_2^2) + \frac{1}{2}X_1^2X_2^2. \quad (4)$$

It is very similar to the quartic oscillator (QO) system with two degrees of freedom, which has been studied extensively both classically and quantum mechanically and is given by [4]

$$H_Q = \frac{1}{2}(\dot{X}_1^2 + \dot{X}_2^2) + \frac{(1-\alpha)}{12}(X_1^4 + X_2^4) + \frac{1}{2}X_1^2X_2^2, \quad (5)$$

where α is a parameter. For $\alpha = 1$ it reduces to Eq.(4). QO system is highly chaotic for $\alpha = 1$ and becomes less chaotic as α decreases as shown in Ref. [4].

When we make a connection between YMCM and QO, we may note that the variables X_i have dimensions of energy (E). H_Q is a constant of motion, say $g^2\varepsilon$, and has dimension of energy density. Hence we normalize all variables by 4th root of $g^2\varepsilon$ (say, E) to make them dimensionless. Thus we get,

$$H' = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{(1-\alpha)}{12}(q_1^4 + q_2^4) + \frac{1}{2}q_1^2 q_2^2, \quad (6)$$

where $H' = 1$, $q_i \equiv X_i/E$ and $\dot{q}_i \equiv \frac{dq_i}{d\tau}$ and $\tau \equiv Et$. Now we can define the distance between two trajectories, $D(\tau)$, in phase space, as

$$D^G(\tau) = \sqrt{\sum_{i=1}^2 \left((q_i - q'_i)^2 + (\dot{q}_i - \dot{q}'_i)^2 \right)}, \quad (7)$$

where primed and unprimed variables describe two different trajectories. Superscript G refers to the fact that it is the general definition used in text books [6] and Ref. [4]. Note that without normalized variables, $D(t)$ is not dimensionally correct. Now the maximum Lyapunov exponent is defined as

$$\lambda_1 \equiv \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left| \frac{D(\tau)}{D(0)} \right| = \frac{\lambda_E}{E}, \quad (8)$$

where

$$\lambda_E \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left| \frac{D(t, E)}{D(0, E)} \right|. \quad (9)$$

Here $D(t, E)$ is given by

$$D(t, E) = \frac{1}{E} \sqrt{\sum_{i=1}^2 \left((X_i - X'_i)^2 + (\dot{X}_i - \dot{X}'_i)^2 / E^2 \right)}, \quad (10)$$

λ_E is equal to λ_1 for $E = 1$ and for arbitrary E or $g^2\varepsilon$, λ_E scales with E or the 4th root of ε which has the dimension of energy. This is similar to the results of Müller *et al.*, obtained numerically, whereas here it follows from a dimensional arguments. We have also numerically verified Eq. (9) with our model; this is tabulated in Table 1.

The maximum Lyapunov exponent is evaluated using the procedure of Joy and Sabir [4] using the Hamiltonian, Eq. (5), with $\alpha = 0.99$, which is close to 1. For α exactly equal to 1, numerical calculations are not reliable and large numerical error develops which can be seen from the numerical values of ε , which is a constant of motion. The Hamilton equations of motion, to be solved numerically, are

$$\begin{aligned}\dot{X}_1 &= X_3 ; \dot{X}_3 = (\alpha - 1)X_1^3/3 - X_1X_2^2 ; \\ \dot{X}_2 &= X_4 ; \dot{X}_4 = (\alpha - 1)X_2^3/3 - X_2X_1^2 ,\end{aligned}\tag{11}$$

which follow from the Eq. (5). These equations are to be evolved along with the equation for their variations $Y_i \equiv \delta X_i$, which are given by

$$\begin{aligned}\dot{Y}_1 &= Y_3 ; \dot{Y}_3 = ((\alpha - 1)X_1^2 - X_2^2)Y_1 - 2X_1X_2Y_2 ; \\ \dot{Y}_2 &= Y_4 ; \dot{Y}_4 = ((\alpha - 1)X_2^2 - X_1^2)Y_2 - 2X_1X_2Y_1 .\end{aligned}\tag{12}$$

Then λ_E is evaluated using the Eq. (9) for different E and is tabulated in Table 1. It is in good agreement with the scaling relation.

However, it should be noted that the procedure used by Müller to evaluate λ is not reliable here. They obtained λ from the slope of the log-plot of $D(t)$ Vs t . But here the slope depends on the initial conditions one uses. As an example a plot is given in Fig. 1, where two trajectories, separated initially

by small distance, are evolved using the Eq. (11) for $\alpha = 0.99$ with different initial conditions but with same energy ($E = 1$).

Generally in gases and liquids, thermalization is due to collisions and the ratio of Kolmogorov entropy (KS-entropy) to collision frequency is related to the change in thermodynamic entropy S [7]. In our case there are no collisions but the nonlinearity of Yang-Mills equations drives the system chaotic or ergodic and hence the thermalization. KS-entropy is related to the sum of positive Lyapunov exponents and here we have only one positive LE and hence $S_{KS} = \lambda_E$. Let t_{th} is the thermalization time, then $S_{KS}t_{th} \approx S$ or

$$t_{th} \approx S/S_{KS} = S(E)/\lambda_E \propto \frac{1}{\lambda_1 E}. \quad (13)$$

If we assume that our system is a subsystem and is in equilibrium with a larger system with average energy density (ε) propotional to the $4th$ power of temperature (T), then t_{th} is inversely propotional to T ; this is consistent with earlier $(3 + 1)$ - dimensional calculations [2, 3]. In addition there is a logarithmic dependence on T because of S , which depends on energy density ($S \propto \log(\varepsilon)$) as discussed in Ref. [5].

$(3 + 1)$ - dimensional YM system, discussed in Ref. [2, 3], is a system with large degrees of freedom due to spatial dependence, and the thermalization there occurs, due to equal energy sharing between various modes by nonlinear interactions. In our case, we have Yang-Mills fields at a point and energy sharing is due to their nonlinear interactions between only 2 degrees of freedom. It should be noted that recent studies of systems with fewer degrees of freedom (even two degrees of freedom), which are chaotic systems, do exhibits thermalization or equipartitioning of energy and other statistical properties as discussed in Ref. [5] and reference there in. Here it is chaos

which plays the role of collisions in statistical mechanics. In statistical mechanics, indeed we need large degrees of freedom to have a chaotic or ergotic motion. But here, even two degrees of freedom exhibits chaos and hence almost ergotic because of nonlinearity in the system. To understand it further, we may rewrite the Hamiltonian, Eq. (3), as

$$H_2 = \frac{1}{2}(E_1^2 + E_2^2) + \frac{1}{2}B_3^2, \quad (14)$$

using the definition of electric and magnetic fields in terms of field tensor

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc}A_b^\mu A_c^\nu,$$

a, b, c are color indices which take values 1, 2, 3 and Lorentz indices $\mu, \nu = 0, 1, 2, 3$ with metric $(1, -1, -1, -1)$. ϵ_{abc} is antisymmetric Levi-Civita tensor. In our case $a = 1, 2$ and we use hedge hog ansatz and hence we have electric fields E_1, E_2 and magnetic field B_3 . Now thermalization means sharing energy between electric and magnetic fields. If we start with whole energy, say, in electric field E_1 , after thermalization the energy will be equally distributed between E_1, E_2 and B_3 . As we discussed in [5], whenever a system is almost chaotic equipartition of energy takes place and

$$\left\langle \dot{x}_1 \frac{\partial H_2}{\partial \dot{x}_1} \right\rangle = \left\langle \dot{x}_2 \frac{\partial H_2}{\partial \dot{x}_2} \right\rangle = \left\langle x_1 \frac{\partial H_2}{\partial x_1} \right\rangle = \left\langle x_2 \frac{\partial H_2}{\partial x_2} \right\rangle, \quad (15)$$

which is same as

$$\langle E_1^2 \rangle = \langle E_2^2 \rangle = \langle B_3^2 \rangle = \frac{2}{3}\varepsilon, \quad (16)$$

Here the angular brackets indicate the time average, which is also equal to the phase space average by ergodic theorem. Hence t_{th} gives the time required to equilibrate energy between electric and magnetic fields and their components. This is a very important property of YM fields which, for example

in relativistic heavy ion collisions, will thermalize the coherent fields formed immediately after the collision. This is due to the relaxation phenomena associated with color degrees of freedom rather than momentum relaxation. Further more, if we extrapolate the results of Ref. [5] on the quartic oscillators, with large degrees of freedom, to our system with large color degrees of freedom or large number of YMCM systems with few color degrees of freedom, the energy distribution in any one component of fields may be exponential decay with decay constant inversely proportional to temperature. In other words, corresponding field distribution is Gaussian with it's width proportional to temperature. This is similar to the case of momentum relaxation, where one gets Gaussian momentum distribution of particles in statistical mechanics. It implies that, at low temperature, the probability to find any field component with low energy is high and hence the large number of gluons are with low energy. That is, low energy gluons macroscopically (largely) occupy YM system at low temperature which may be related to confinement. But at high temperature, field components with higher energy increase, but those of low energy decrease. Finally the probability to find low energy and high energy gluons will be of the same order. Now YM system may be in deconfined state.

So far our discussion is based on λ which is evaluated using Eq. (10). However, in Ref. [3], they used somewhat different definition for $D(t)$. They defined

$$D^M(t) = |\sum_i (E_i^2 - E_i'^2)|, \quad (17)$$

where E_i are the electric field components, which reduces (in our notations)

to

$$D^M(t) = |\sum_i (\dot{X}_i^2 - \dot{X}'_i{}^2)|/g = 2|\sum_i \dot{Y}_i \dot{X}_i|/g. \quad (18)$$

\dot{X}_i and \dot{Y}_i are evaluated from the Eqs. (11) and (12). Even though $D^M(t)$ is different from $D^G(t)$, we get the same results. If we take square root instead of modulus in Eq. (18) we get λ_E about half of λ_E obtained from Eq. (10). This is clear from the expression for $D^M(t)$ where the exponentially diverging term Y_i appears as linear term, whereas in $D^G(t)$ it is quadratic.

In our numerical calculations, as can be seen from the Table 1, theoretical scaling behaviour of λ with E is confirmed. We also point out that the evaluation of λ by the slope measurement of log plot of $D(t)$ Vs t , where $D(t)$ is evaluated from two independent nearby trajectories, is not reliable. Where as standard method given in text books gives reliable results. This is probably because of the fact that LE concept is at the linear perturbation level, Eq. (12), and not with full nonlinear variations, as in Ref. [3] (by following two nearby trajectories separately).

In conclusion, using a simple QO model to represent the homogeneous $SU(2)$ YM fields, many qualitative properties obtained by extensive numerical calculation of $(3+1)$ - dimensional YM fields can be reproduced. We obtained the linear relationship between maximum Lyapunov exponent and the 4th root of energy density, E , numerically as well as from a dimensional arguments. The slope of the linear relationship is 0.4 which is of the same order as that of Müller *et al.*. We also obtained an expression for thermalization time and explained the physical picture of thermalization in our model. The thermalization time is of the order of $1/(\lambda_1 E) \approx 1 fm/c$ for $g = 1$ and the energy density $\varepsilon = 1 GeV/fm^3$ and inversely depends on the 4th power

of energy density. In the case of relativistic heavy ion collisions, it may give an estimate of the order of magnitude of time required to redistribute the energy, stored initially in coherent fields, among all degrees of freedom associated with spatial, color, electric and magnetic fields, etc.

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Figure Caption

Figure 1: Plots of $\ln \left| \frac{D(t,E)}{D(0,E)} \right|$ as a function of t for $\alpha = 0.99$, $E = 1$ with initial conditions $X_3 = .01$ (curve 1), 0.1 (curve 2), 0.4 (curve 3), 1.0 (curve 4), 1.393 (curve 5) respectively with all other $X_i = 0$.

Table 1

$g^2\varepsilon$	1	16	81	256	625
λ_E	0.4	0.78	1.17	1.56	1.95

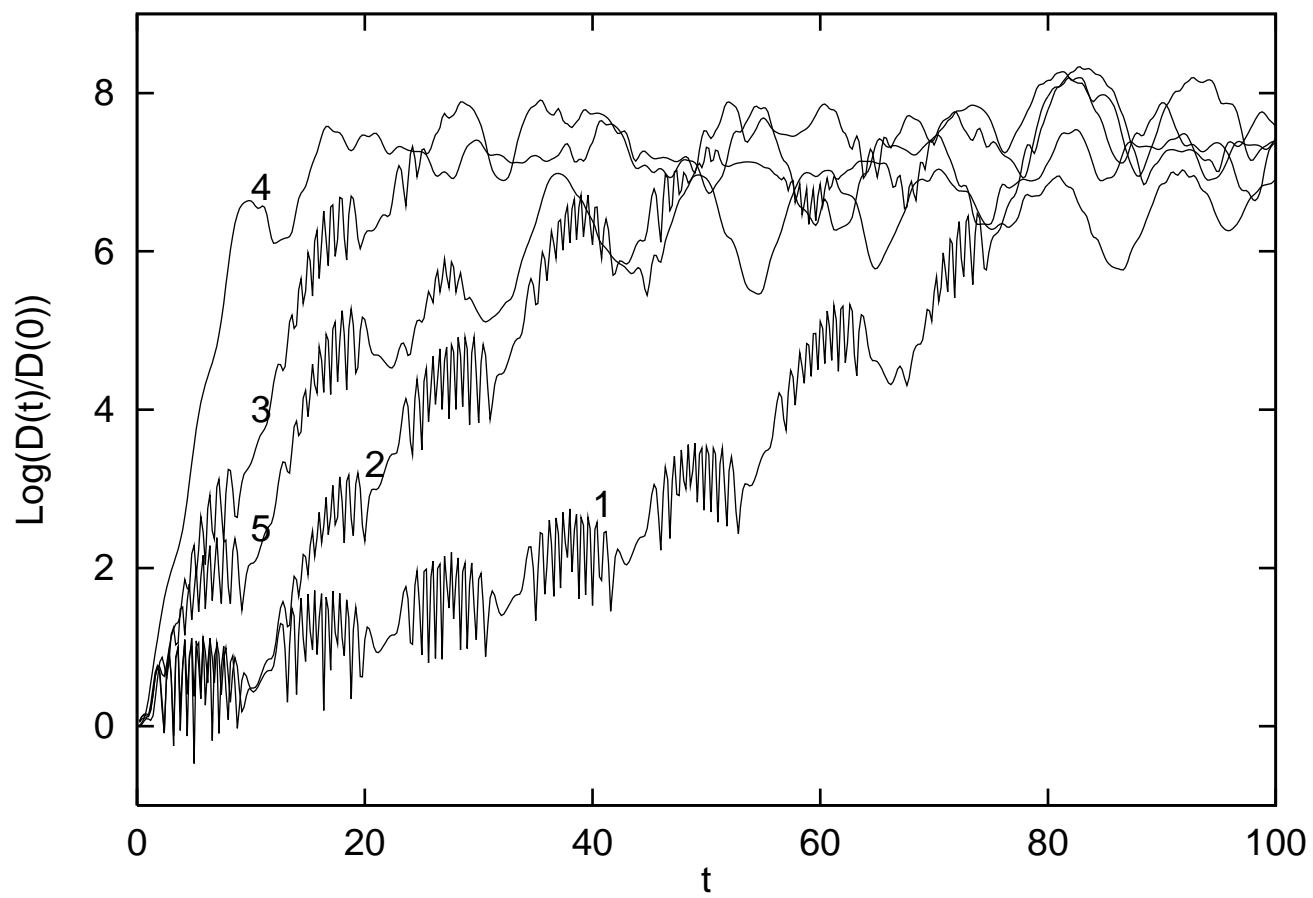


Fig.1